

# ROTATION NUMBERS FOR QUASIPERIODICALLY FORCED CIRCLE MAPS – MODE-LOCKING VS STRICT MONOTONICITY

Kristian Bjerklöv\* and Tobias Jäger†

2nd February 2008

## Abstract

We describe the relation between the dynamical properties of a quasiperiodically forced orientation-preserving circle homeomorphism  $f$  and the behavior of the fibered rotation number w.r.t. strictly monotone perturbations. Despite the fact that the dynamics in the forced case can be considerably more complicated, the result we obtain is in perfect analogy with the one-dimensional situation. In particular, the fibered rotation number behaves strictly monotonically whenever the rotation vector of  $f$  is irrational. This answers a question posed by Herman [1].

## 1 Introduction

For an orientation-preserving circle homeomorphism  $g : \mathbb{T}^1 \rightarrow \mathbb{T}^1$  it is well-known that the rotation number behaves strictly monotonically whenever  $g$  has no periodic points. On the other hand, the only situation where mode-locking occurs (i.e. the rotation number is stable w.r.t. perturbations) is when there exists a closed interval which is mapped inside its own interior by some iterate of  $g$ , as in the case of a stable periodic orbit. If the rotation number only stays constant on one side this corresponds to the existence of parabolic periodic points. In this paper we show that exactly the same picture holds for quasiperiodically forced (qpf) orientation-preserving circle homeomorphisms, where  $p, q$ -invariant strips (as introduced in [2] and [3]) serve as natural analogues of periodic orbits.

We consider continuous maps  $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$  of the form

$$(1.1) \quad f(\theta, x) = (\theta + \omega, f_\theta(x)) ,$$

where  $\omega \in [0, 1] \setminus \mathbb{Q}$ . In addition, we require all *fiber maps*  $f_\theta$  to be orientation-preserving circle homeomorphisms. The class of all such maps will be denoted by  $\mathcal{F}$ . Further, we will use the notation  $f_\theta^n(x) = \pi_2 \circ f^n(\theta, x)$ . Given any continuous lift  $F : \mathbb{T}^1 \times \mathbb{R} \rightarrow \mathbb{T}^1 \times \mathbb{R}$  of  $f \in \mathcal{F}$ , the limit

$$(1.2) \quad \rho(F) := \lim_{n \rightarrow \infty} (F_\theta^n(x) - x)/n$$

exists and is independent of  $\theta$  and  $x$  [1]. The *fibered rotation number* of  $f$  is defined as  $\rho(f) := \rho(F) \bmod 1$ . In order to state our results, we need the following notions:

**Definition 1.1.** Suppose  $f$  is a qpf circle homeomorphism with lift  $F$ . We let

$$F_\varepsilon(\theta, x) := (\theta + \omega, F_\theta(x) + \varepsilon)$$

and say the rotation number is strictly monotone in  $f$  if the map  $\varepsilon \mapsto \rho(F_\varepsilon)$  is strictly monotone at  $\varepsilon = 0$ . If  $\varepsilon \mapsto \rho(F_\varepsilon)$  is constant in a neighborhood of  $\varepsilon = 0$  we say  $f$  is mode-locked.

**Theorem A.** Suppose  $f \in \mathcal{F}$  has no  $p, q$ -invariant strip. Then the rotation number is strictly monotone in  $f$ . In particular, this holds whenever  $\omega$ ,  $\rho(f)$  and 1 are rationally independent.<sup>1</sup>

Note that the second statement in Theorem A is the answer to the question asked by Herman in [1, Section 5.18, p. 497].

**Theorem B.** Suppose  $f \in \mathcal{F}$  is mode-locked. Then there exists a closed annulus, bounded by continuous curves, which is mapped into its own interior by some iterate of  $f$ .

For a description of the critical cases with one-sided monotonicity, we refer to Section 4.

\*Department of Mathematics, University of Toronto, Canada. Email: bjerklöv@math.utoronto.ca

†Mathematisches Institut, Universität Erlangen-Nürnberg, Germany. Email: jaeger@mi.uni-erlangen.de

<sup>1</sup>The presence of  $p, q$ -invariant graphs forces  $\omega$ ,  $\rho(f)$  and 1 to be rationally dependent, see Theorem 1. Hence the second statement is an immediate consequence of the first.

There is one well-known situation where a similar result to that in Theorem A holds, namely in the case of the (generalized) Harper map

$$(1.3) \quad s_E : \mathbb{T}^1 \times \overline{\mathbb{R}} \rightarrow \mathbb{T}^1 \times \overline{\mathbb{R}}, \quad (\theta, x) \mapsto \left( \theta + \omega, V(\theta) - E - \frac{1}{x} \right),$$

where  $V : \mathbb{T}^1 \rightarrow \mathbb{R}$  is a continuous function and  $E \in \mathbb{R}$ . By letting  $x_n = u_{n+1}/u_n$ , there is a 1-1 correspondence between orbits of this map and formal eigenfunctions of the discrete Schrödinger operator  $(H_\theta u)_n = -(u_{n+1} + u_{n-1}) + V(\theta + n\omega)u_n$ . Note that, by identifying  $\overline{\mathbb{R}}$  with  $\mathbb{T}^1$ , (1.3) defines an element of  $\mathcal{F}$  and we can therefore speak of the fibered rotation number of  $s_E$ . Due to a fruitful interplay between spectral theory and dynamical systems methods it is well-known that the function  $E \mapsto \rho(s_E)$  is monotone and continuous, and maps  $\mathbb{R}$  onto  $[0, 1]$ . In fact, it can be shown that  $\rho(s_E) = k(E)$ , where  $k(E)$  denotes the integrated density of states for the corresponding Schrödinger operator. Moreover, if  $\rho(s_E)$  is constant on an interval, then  $\rho(s_E) = k\omega \bmod 1$  for some integer  $k$  [4, 5]. In particular, if  $\rho(s_E)$  and  $\omega$  are rationally independent, then  $\rho(s_E)$  is strictly monotone in  $E$ . We would like to stress that analogue results first where established for the continuous Schrödinger equation with almost periodic potentials in the fundamental paper [5].

In contrast to the situation for the Harper map and despite a large number of numerical studies ([6] gives a good overview and further reference), there are hardly any rigorous results about other parameter families of qpf circle maps. The most prominent example is probably the qpf Arnold circle map

$$f_{\alpha, \beta, \tau} : \mathbb{T}^2 \mapsto \mathbb{T}^2, \quad (\theta, x) \mapsto \left( \theta + \omega, x + \tau + \frac{\alpha}{2\pi} \sin(2\pi x) + \beta g(\theta) \bmod 1 \right)$$

with parameters  $\alpha, \tau \in [0, 1]$ ,  $\beta \in \mathbb{R}$  and continuous forcing function  $g : \mathbb{T}^1 \rightarrow \mathbb{R}$ . This map was first studied in [7], where it was proposed as a simple model of an oscillator forced at two incommensurate frequencies. It was observed that the fibered rotation number seems to stay constant on open regions in the  $(\alpha, \tau)$ -parameter space, the so-called Arnold tongues (see also [8, 9, 10]).<sup>2</sup> Although this was clearly expected and backed up by numerical evidence, to the knowledge of the authors it was not known so far that the rotation numbers have to be rationally related inside these tongues. Now this is certainly a direct consequence of Theorem A, which more generally implies the following

**Corollary A.** *Suppose  $(f_\tau)_{\tau \in \mathbb{R}}$  is a parameter family in  $\mathcal{F}$  with lifts  $(F_\tau)_{\tau \in \mathbb{R}}$ . Further assume  $F_\tau$  depends continuously on  $\tau$  and  $F_{\tau, \theta}(x) < F_{\tau', \theta}(x) \forall (\theta, x)$  whenever  $\tau < \tau'$ . Then  $\tau \mapsto \rho(f_\tau)$  can only stay constant in  $\tau_0$  if  $\omega$ ,  $\rho(f_{\tau_0})$  and 1 are rationally dependent.*

Finally, we note that Theorem A can also give interesting information in situations with rationally related rotation numbers: The simulations in [8, 9] suggest that for certain parameters  $\alpha, \beta$  the  $\tau$ -interval with fibered rotation number zero is collapsed to a single point. This observation will be confirmed in a forthcoming paper [11], by showing that for suitable forcing function  $g$  and parameters  $\alpha, \beta$  the map  $f_{\alpha, \beta, 0}$  has fibered rotation number zero, but minimal dynamics and therefore no  $p, q$ -invariant strips. Consequently  $\tau \mapsto \rho(f_{\alpha, \beta, \tau})$  is strictly monotone in  $\tau = 0$  by Theorem A.

**Acknowledgments.** This cooperation started during a workshop on ‘Dynamics of cocycles and one-dimensional spectral theory’ at Oberwolfach 2005. We would like to thank the organizers David Damanik, Russell Johnson and Daniel Lenz for making this possible. K.B. was partially supported by SVEFUM. T.J. was supported by grant Ke 514/6-1 of the German Science Foundation (DFG), which also funded a visit of K.B. to Erlangen.

## 2 Preliminaries

For any set  $A \subseteq \mathbb{T}^2$  and  $\theta \in \mathbb{T}^1$  we let  $A_\theta := \{x \in \mathbb{T}^1 \mid (\theta, x) \in A\}$ . In the simplest case  $p = q = 1$ , the definition of an invariant strip is as follows:

<sup>2</sup>For the unforced Arnold circle map this is certainly well-known.

**Definition 2.1.** Let  $f \in \mathcal{F}$ .  $A \subseteq \mathbb{T}^2$  is called a  $1, 1$ -invariant strip if it is compact,  $f$ -invariant and for all  $\theta \in \mathbb{T}^1$  the set  $A_\theta$  consists of exactly one nontrivial interval (i.e.  $\neq \emptyset$  or  $\mathbb{T}^1$ ).

Note that in particular this definition includes continuous invariant curves. As the formulation for the general case  $(p, q) \in \mathbb{N}^2$  is slightly technical and we actually do not have to use it, we refrain from giving the precise definition here and refer to [2] or [3]. However, in order to get a basic idea one should think of  $p$   $p$ -periodic closed curves which are permuted by the action of  $f$  and all wind around the torus  $q$  times in the  $\theta$ -direction. The union of these curves intersects every fiber in exactly in  $pq$  points, and roughly spoken one allows each of these points to be replaced by a closed interval in the definition of a  $p, q$ -invariant strip.

The important thing we have to know here is the fact that the existence of a  $p, q$ -invariant strip  $A$  forces the fibered rotation number to be of the form

$$(2.1) \quad \rho(f) = \frac{k}{q}\omega + \frac{l}{pq} \pmod{1},$$

where the integers  $k, l, p$  and  $q$  are determined by the topological and dynamical structure of  $A$  and vice versa (see Lemma 3.9 in [2]). Further, by going over to a suitable iterate, lifting the system to a  $q$ -fold cover  $(\mathbb{R}/q\mathbb{Z}) \times \mathbb{T}^1$  and rescaling, one can always transform a system with a  $p, q$ -invariant graph into one with a  $1, 1$ -invariant graph (compare Lemma 2.15 in [3]; this is the reason why we do not need the general definition). For the latter (2.1) implies that the fibered rotation number will be of the form  $\rho(f) = k\omega \pmod{1}$ , and by conjugating with  $h : (\theta, x) \mapsto (\theta, x - k\theta)$  we can finally assume that  $\rho(f) = 0$ .

The proof of Theorem A is based on a classification result for qpf circle homeomorphisms obtained in [3]. In order to state it we have to introduce the ‘*deviations from the constant rotation*’, which are defined as

$$(2.2) \quad D_F(n, \theta, x) := F_\theta^n(x) - x - n\rho(F).$$

In contrast to the unforced situation, these deviations do not have to be uniformly bounded in  $(n, \theta, x)$  in the forced case. However, if the deviations are unbounded, they have to be unbounded on every single orbit (see Theorem 1.3 in [10]). This motivates the following definition:

**Definition 2.2.**  $f \in \mathcal{F}$  is called  $\rho$ -bounded if the deviations are uniformly bounded and  $\rho$ -unbounded otherwise.

Note that the boundedness of the deviations does not depend on the choice of the lift  $F$ . We also remark that, again by Theorem 1.3 in [10], in the  $\rho$ -unbounded case there always exist orbits for which the deviations are unbounded from above (and similarly from below). It turns out that in the  $\rho$ -bounded case, the natural analogue of the Poincaré Classification Theorem (e.g. [12]) holds:

**Theorem 1 (Theorems 3.1 and 4.1 in [3]).** If  $f \in \mathcal{F}$  is  $\rho$ -bounded, then either there exists a  $p, q$ -invariant strip and  $\rho(f)$ ,  $\omega$  and  $1$  are rationally dependent or  $f$  is semi-conjugate to the irrational torus translation  $(\theta, x) \mapsto (\theta + \omega, x + \rho(f))$  by a semi-conjugacy  $h$  which is fiber-respecting (i.e.  $\pi_1 \circ h = \pi_1$ ). If  $f \in \mathcal{F}$  is  $\rho$ -unbounded, then neither of these alternatives can occur and the map is always topologically transitive.

Examples of systems with  $\rho$ -unbounded behavior can be found e.g. in [1], similar examples with minimal dynamics in [13].

For the proof of Theorem B we have to introduce a number of notions and facts concerning qpf monotone maps. We call  $F : \mathbb{T}^1 \times \mathbb{R} \rightarrow \mathbb{T}^1 \times \mathbb{R}$  a *qpf monotone map* if it is continuous, has skew product structure as in (1.1) and all fiber maps  $F_\theta : \mathbb{R} \rightarrow \mathbb{R}$  are monotonically increasing. In particular, this is true if  $F$  is the lift of some  $f \in \mathcal{F}$ . Similar to Definition 2.1 we define an  $(F)$ -invariant strip as a compact  $F$ -invariant set which consists of exactly one non-empty

interval on every fiber. The *upper* and *lower bounding graphs* of an invariant strip  $A$  are defined as

$$(2.3) \quad \varphi_A^+(\theta) := \sup A_\theta \quad \text{and} \quad \varphi_A^-(\theta) := \inf A_\theta,$$

respectively. Due to compactness,  $\varphi_A^+$  will be upper semi-continuous (u.s.c.) and  $\varphi_A^-$  lower semi-continuous (l.s.c.). Note that more generally we can apply definition (2.3) to any bounded  $F$ -invariant set  $A$  in order to obtain invariant (but not necessarily semi-continuous) bounding graphs. If  $A$  and  $B$  are two invariant strips, we use the notation

$$A \preceq B \quad :\Leftrightarrow \quad \varphi_A^- \leq \varphi_B^- \quad \text{and} \quad \varphi_A^+ \leq \varphi_B^+$$

and

$$A \prec B \quad :\Leftrightarrow \quad \varphi_A^+ < \varphi_B^-.$$

Further, for any two graphs  $\varphi, \psi : \mathbb{T}^1 \rightarrow \mathbb{R}$  with  $\varphi \leq \psi$  we let

$$[\varphi, \psi] := \{(\theta, x) \mid x \in [\varphi(\theta), \psi(\theta)]\},$$

similarly for open and half-open intervals. For any graph  $\varphi : \mathbb{T}^1 \rightarrow \mathbb{R}$  we denote its point set by the corresponding capital letter, i.e.  $\Phi := \{(\theta, \varphi(\theta)) \mid \theta \in \mathbb{T}^1\}$ , likewise for curves  $\gamma : I \rightarrow \mathbb{R}$  which are only defined on a subinterval  $I \subseteq \mathbb{T}^1$ . Further, we let

$$(2.4) \quad \varphi^+ := \varphi_\Phi^+ \quad \text{and} \quad \varphi^- := \varphi_\Phi^-.$$

For simplicity, we denote  $\varphi^{+-} = (\varphi^+)^-$ ,  $\varphi^{-+} = (\varphi^-)^+$ .

We call a  $F$ -invariant strip  $A$  *minimal* whenever it does not strictly contain any smaller  $F$ -invariant strip. Minimality of  $A$  is equivalent to the fact that  $\varphi_A^{+-} = \varphi_A^-$  and  $\varphi_A^{-+} = \varphi_A^+$  (see [3]). The following lemma describes a simple procedure to obtain minimal strips from semi-continuous invariant graphs (see Lemma 2.5 together with Definition 2.7 and Remark 2.8(a) in [3]):

**Lemma 2.3.** *Suppose  $F$  is a qpf monotone map and  $\varphi$  is an u.s.c. invariant graph. Then  $[\varphi^-, \varphi^{-+}]$  is a minimal invariant strip. Similarly, if  $\varphi$  is l.s.c. then  $[\varphi^{+-}, \varphi^+]$  is a minimal invariant strip.*

One of the most important properties of a minimal strip  $A$  is the fact that it is *pinched*, meaning that there is at least one fiber which intersects  $A$  only in one single point.<sup>3</sup> This follows from

**Lemma 2.4 (Theorem 4.5 in [14]).** *Suppose  $\varphi$  is an upper semi-continuous invariant graph of a qpf monotone map. Then  $[\varphi^-, \varphi]$  is pinched. Similarly, if  $\varphi$  is lower semi-continuous then  $[\varphi, \varphi^+]$  is pinched.*

### 3 Strict monotonicity: Proof of Theorem A

From now on, we fix  $f \in \mathcal{F}$  with lift  $F_0$  and let  $F_\varepsilon$  be as in Definition 1.1. Theorem 1 allows to divide the problem into two cases, namely the  $\rho$ -unbounded one and the one with a semi-conjugacy to an irrational translation. We treat them separately in the following two lemmas, starting with the  $\rho$ -unbounded case:

**Lemma 3.1.** *Suppose  $f$  is  $\rho$ -unbounded. Then the rotation number is strictly monotone in  $f$ .*

---

<sup>3</sup>Obviously the invariance implies that the set  $\mathcal{P}_A := \{\theta \in \mathbb{T}^1 \mid \text{card}(A_\theta) = 1\}$  is dense in  $\mathbb{T}^1$  in this case, and from Baire's Theorem it follows that  $\mathcal{P}_A$  is even residual [14].

*Proof.* We only have to show that for any  $\varepsilon > 0$  there holds  $\rho(F_\varepsilon) > \rho(F_0)$ , as the case where  $\varepsilon < 0$  is symmetric. Hence, fix any  $\varepsilon > 0$ . Due to uniform continuity, there exists  $\delta > 0$  such that  $|\theta - \theta'| < \delta$  implies  $F_{0,\theta}(x) \leq F_{\varepsilon,\theta'}(x) \forall x \in \mathbb{R}$ . By induction and using the monotonicity of the fiber maps we obtain

$$(3.1) \quad F_{0,\theta}^n(x) \leq F_{\varepsilon,\theta'}^n(x) \quad \forall \theta, \theta' : |\theta - \theta'| < \delta \quad \forall x \in \mathbb{R}, n \in \mathbb{N}.$$

Fix  $N \in \mathbb{N}$  such that  $\{n\omega\}_{n=0}^N$  is  $\delta$ -dense in the circle. As we mentioned above, in the  $\rho$ -unbounded case there always exists an orbit  $(\theta_0, x_0)$  on which the deviations are unbounded from above. Let  $\theta_k := \theta_0 + k\omega$ . As all maps  $F_\theta^n$  are lifts of circle homeomorphisms, obviously there holds

$$(3.2) \quad |D(n, \theta, x) - D(n, \theta, x')| \leq 1 \quad \forall n \in \mathbb{N}, \theta \in \mathbb{T}^1, x, x' \in \mathbb{R}.$$

Further, there exists a constant  $C > 0$  which satisfies

$$(3.3) \quad |D(n, \theta, x) - D(n + m, \theta, x)| \leq C \quad \forall n \in \mathbb{N}, \theta \in \mathbb{T}^1, x \in \mathbb{R} \text{ and } m = 1, \dots, N.$$

As the deviations of the orbit of  $(\theta_0, x_0)$  are unbounded from above, we can choose  $M \in \mathbb{N}$  with  $D(M, \theta_0, x_0) \geq C + 2$ . Together with (3.3) this implies

$$(3.4) \quad D(M, \theta_k, x) \geq 1 \quad \forall k = 0, \dots, N, x \in \mathbb{R}.$$

Now we proceed inductively to show that for any  $\theta \in \mathbb{T}^1, x \in \mathbb{R}$  there holds

$$(3.5) \quad F_{\varepsilon,\theta}^{nM}(x) - x \geq n(M\rho(F_0) + 1) \quad \forall n \geq 0,$$

which immediately implies  $\rho(F_\varepsilon) \geq \rho(F_0) + \frac{1}{M}$ . For  $n = 0$  there is nothing to prove, so assume the statement holds for some  $n \geq 0$ . Let  $x' := F_{\varepsilon,\theta}^{nM}(x)$  and choose  $k \in \{0, \dots, N\}$  such that  $|\theta + nM\omega - \theta_k| < \delta$  (recall the choice of  $N$ ). Then

$$\begin{aligned} F_{\varepsilon,\theta}^{(n+1)M}(x) - x &= F_{\varepsilon,\theta+nM\omega}^M(x') - x' + x' - x \\ &\stackrel{(3.1)}{\geq} F_{0,\theta_k}^M(x') - x' + n(M\rho(F_0) + 1) \\ &= M\rho(F_0) + D(M, \theta_k, x') + n(M\rho(F_0) + 1) \\ &\stackrel{(3.4)}{\geq} (n+1)(M\rho(F_0) + 1). \end{aligned}$$

□

This only leaves the case where  $f$  is semi-conjugate to an irrational torus translation. In order to treat this, it is convenient to look at invariant measures. Let  $h : (\theta, x) \mapsto (\theta, h_\theta(x))$  be the fiber-respecting semi-conjugacy from Theorem 1 and define a measure  $\mu$  on  $\mathbb{T}^2$  by  $\mu(A) = \lambda^2(h(A)) \forall A \in \mathcal{B}(\mathbb{T}^2)$ , where  $\lambda^n$  denotes the Lebesgue measure on  $\mathbb{T}^n$ . Then it is easy to see that  $\mu$  is an ergodic  $f$ -invariant probability measure. (In fact, it can be deduced from Theorem 4.1 in [15] that  $\mu$  is the only  $f$ -invariant probability measure in this situation.) From the definition of  $\mu$  it follows that

$$(3.6) \quad \mu(A) = \int_{\mathbb{T}^1} \mu_\theta(A_\theta) d\theta \quad \forall A \in \mathcal{B}(\mathbb{T}^2)$$

where  $A_\theta := \{x \in \mathbb{T}^1 \mid (\theta, x) \in A\}$  and the so-called fiber measures (or conditional measures) are defined as  $\mu_\theta(B) = \lambda^1(h_\theta(B)) \forall B \in \mathcal{B}(\mathbb{T}^1)$ . Obviously, by this definition the measures  $\mu_\theta$  are all continuous (in the sense that  $\mu_\theta(\{x\}) = 0 \forall x \in \mathbb{T}^1$ ) and  $f_\theta$  maps  $\mu_\theta$  to  $\mu_{\theta+\omega}$ , i.e.  $\mu_{\theta+\omega} = f_\theta^* \mu_\theta$ . (A general discussion of fiber measures can for example be found in [16].) Therefore the strict monotonicity of the rotation number in the semi-conjugated case is a consequence of the following lemma.

**Lemma 3.2.** *Suppose  $f$  has an ergodic invariant probability measure  $\mu$  with continuous fiber measures  $\mu_\theta$ . Then the rotation number is strictly monotone in  $f$ . In particular, this is true whenever  $f$  is semi-conjugate to an irrational translation of the torus.*

*Proof.* Fix  $\varepsilon > 0$ . We identify  $\mu$  and  $\mu_\theta$  with their natural lifts to  $\mathbb{T}^1 \times \mathbb{R}$  and  $\mathbb{R}$ , respectively. As the fiberwise rotation number does not depend on  $(\theta, x)$ , it suffices to show that there exist one  $(\theta, x) \in \mathbb{T}^1 \times \mathbb{R}$  such that

$$(3.7) \quad \lim_{n \rightarrow \infty} \frac{1}{n} F_{\varepsilon, \theta}^n(x) > \lim_{n \rightarrow \infty} \frac{1}{n} F_{0, \theta}^n(x) .$$

Let us first see that (3.7) is a consequence of the following statement:

$$(3.8) \quad p(\theta) := \min\{p \in \mathbb{N} \mid F_{\varepsilon, \theta}^p(x) \geq F_{0, \theta}^p(x) + 1 \ \forall x \in \mathbb{R}\} < \infty \text{ for } \lambda^1\text{-a.e. } \theta \in \mathbb{T}^1 .$$

Indeed, this implies that for some  $q \in \mathbb{N}$  the set  $A_q := \{\theta \in \mathbb{T}^1 \mid p(\theta) = q\}$  has positive measure, as  $\lambda^1\left(\bigcup_{q \in \mathbb{N}} A_q\right) = 1$ . W.l.o.g. we can assume that  $q = 1$ , otherwise we replace  $f$  by its  $q$ th iterate. Due to the monotonicity and periodicity of the fiber maps we obtain

$$F_{\varepsilon, \theta}^n(x) \geq F_{0, \theta}^n(x) + \sum_{i=0}^{n-1} \mathbf{1}_{A_1}(\theta + i\omega) .$$

As  $\theta \mapsto \theta + \omega$  is ergodic, this implies

$$\lim_{n \rightarrow \infty} \frac{1}{n} F_{\varepsilon, \theta}^n(x) \geq \lim_{n \rightarrow \infty} \frac{1}{n} F_{0, \theta}^n(x) + \lambda^1(A_1) > \lim_{n \rightarrow \infty} \frac{1}{n} F_{0, \theta}^n(x)$$

for  $\lambda^1$ -a.e.  $\theta$ , thus proving (3.7).

It remains to show that the function  $p$  is  $\lambda^1$ -a.s. finite. To that end, note that for any  $\varepsilon > 0$  the function

$$g(\theta, x) := \mu_\theta([x, x + \varepsilon))$$

is  $\mu$ -a.s. strictly positive. Therefore

$$\delta := \frac{1}{2} \int_{\mathbb{T}^2} g \, d\mu > 0 .$$

By Birkhoff's Ergodic Theorem, for  $\mu$ -a.e.  $(\theta, x)$  we have

$$(3.9) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} g \circ f^i(\theta, x) = 2\delta ,$$

and from (3.6) it follows that (3.9) holds  $\mu_\theta$ -a.s. for  $\lambda^1$ -a.e. fixed  $\theta$ . Let  $S \subseteq \mathbb{T}^1$  be a set of measure  $\lambda^1(S) = 1$  which is invariant under rotation by  $\omega$  and has the property that (3.9) holds for  $\mu_\theta$ -a.e.  $x \in \mathbb{T}^1$  whenever  $\theta \in S$ . Then for every  $\theta \in S$  and  $\mu_\theta$ -a.e.  $x$  there holds

$$\kappa(\theta, x) := \min\{k \in \mathbb{N} \mid g \circ f^k(\theta, x) > \delta\} < \infty .$$

We will show that  $p$  only takes finite values on  $S$ . In order to do so, we fix  $\theta_0 \in S$  and  $x_0 \in \mathbb{R}$  such that  $\kappa(\theta_0, x_0) < \infty$ .<sup>4</sup> We will construct sequences  $k_i$  and  $y_i$  ( $i \in \mathbb{N}_0$ ) with the following properties: If  $K_i := \sum_{j=0}^i k_j$ ,  $\theta_i := \theta_0 + K_i \omega$ ,  $x_i := F_{\theta_0}^{K_i}(x_0)$  and  $z_i := F_{\varepsilon, \theta_0}^{K_i}(x_0)$  then

$$(3.10) \quad x_i \leq y_i \leq z_i$$

$$(3.11) \quad \mu_{\theta_i}([x_i, y_i)) \geq i \cdot \delta$$

$$(3.12) \quad \kappa(\theta_i, y_i) < \infty .$$

---

<sup>4</sup>Note that we apply  $g$  and  $\kappa$  also to points  $(\theta, x) \in \mathbb{T}^1 \times \mathbb{R}$  by identifying  $x \in \mathbb{R}$  with its projection to  $\mathbb{T}^1$ .

If  $i > 2/\delta$  then (3.11) gives

$$\mu_{\theta_i}([x_i, z_i]) \geq \mu_{\theta_i}([x_i, y_i]) > 2$$

and consequently, as  $\mu_{\theta}([x, x+1]) = 1 \forall (\theta, x)$  (recall that  $\mu$  is a probability measure)

$$F_{\varepsilon, \theta_0}^{K_i}(x_0) > F_{\theta_0}^{K_i}(x_0) + 2,$$

which in turn implies  $p(\theta_0) \leq K_i$ . As  $\theta_0 \in S$  was arbitrary and  $\lambda^1(S) = 1$ , this shows (3.8).

The sequences  $k_i, y_i$  are constructed as follows: If we let  $y_0 = x_0$  and  $k_0 = 0$ , then for  $i = 0$ , there is nothing to show. Suppose  $k_0, \dots, k_i$  and  $y_0, \dots, y_i$  with the required properties have been chosen. Let  $k_{i+1} := \kappa(\theta_i, y_i)$ . Then

$$\begin{aligned} \mu_{\theta_{i+1}}([x_{i+1}, z_{i+1}]) &\geq \mu_{\theta_{i+1}}([x_{i+1}, F_{\varepsilon, \theta_i}^{k_{i+1}}(y_i)]) \\ &= \mu_{\theta_i}([x_i, y_i]) + \mu_{\theta_{i+1}}([F_{\theta_i}^{k_{i+1}}(y_i), F_{\varepsilon, \theta_i}^{k_{i+1}}(y_i)]) > i\delta + \delta. \end{aligned}$$

Thus we can choose some  $y_{i+1} \in (x_{i+1}, z_{i+1})$  which satisfies (3.11) and (3.12).  $\square$

## 4 Mode-locking: Proof of Theorem B

For  $f \in \mathcal{F}$  with lift  $F_0$  let  $F_\varepsilon$  be as in Definition 1.1. Due to Theorem A, we only have to consider the case where  $f$  has a  $p, q$ -invariant strip. Further, as mentioned in Section 2 we can always assume that  $p = q = 1$  and  $\rho(f) = 0$ . In this situation, the statement of Theorem B is a consequence of the following:

**Proposition 4.1.** *Suppose  $\rho(f) = \rho(F_0) = 0$  and for some  $\varepsilon > 0$  there holds  $\rho(F_\varepsilon) = 0$ . Then there exists a continuous curve  $\gamma^+ : \mathbb{T}^1 \rightarrow \mathbb{R}$  which is mapped strictly below itself by some iterate of  $F_0$ . Similarly, if  $\rho(F_{-\varepsilon}) = 0$  then there exists a curve  $\gamma^-$  which is mapped strictly above itself by some iterate of  $F_0$ .*

For if  $f$  is mode-locked, then we obtain two curves  $\gamma^+$  and  $\gamma^-$  which are mapped strictly above, respectively below themselves by some iterate of  $F_0$ . It is easy to see that  $\gamma^+$  and  $\gamma^-$  must be disjoint and that they are lifts of the continuous boundaries of a closed annulus  $\mathcal{A} \subseteq \mathbb{T}^2$  which is mapped inside its own interior by some iterate of  $F_0$ .

**Remark 4.2.** *As we reduced the problem to the situation with fibered rotation number 0, the closed annulus from Proposition 4.1 will be of ‘homotopy type’  $(1, 0)$ , meaning that it only winds once around the torus, in the  $\theta$ -direction. In the general case, by redoing the transformations described in Section 2, we obtain a closed annulus of homotopy type  $(q, k)$  with  $q$  and  $k$  as in (2.1).*

Proposition 4.1 also contains the description of the one-sided cases: Suppose the rotation number only remains constant on one side, for example  $\rho(F_\varepsilon) = 0$  for some  $\varepsilon > 0$  but  $\rho(F_{\varepsilon'}) < 0 \forall \varepsilon' < 0$ . Then the proposition still yields the existence of a closed curve  $\gamma^+$  which is mapped below itself by some iterate of  $F_0$ . The iterates of  $\gamma^+$  form a monotonically decreasing sequence of continuous curves. In the limit, they have to converge to an upper semi-continuous graph  $\varphi^+$  which is the upper boundary of an invariant strip. Projecting the situation to the torus, we obtain the existence of a closed curve which is mapped towards some  $1, 1$ -invariant strip in the clockwise direction. (Note that ‘being mapped above or below itself’ does not make any sense on the torus, unless there is some kind of reference object.) As there is no curve which is mapped towards this strip in the counterclockwise direction, the situation resembles the one with a parabolic periodic orbit in the one-dimensional case. Again, this picture remains valid in the case of a general  $p, q$ -invariant strip, only the homotopy type of all objects involved changes.

Before we turn to the proof of Proposition 4.1, we need two auxiliary lemmas. The first one concerns the possible dynamics between two neighboring minimal invariant strips. As the proof is basically the same as in [2, Part 2 of the proof of Thm. 4.4], we keep it rather brief and refer to [2] for more details.

**Lemma 4.3.** *Suppose  $A \prec B$  are two minimal  $F_0$ -invariant strips and that there is no invariant strip strictly in between. Let  $C := (\varphi_A^+, \varphi_B^-)$ . Then either  $F_{0|C}$  is topologically transitive or there exists a wandering open set  $W \subseteq C$ .*

*Proof.* Suppose neither of these two alternatives holds. Then, as  $F_{0|C}$  is not transitive, there exist open balls  $U, V$  such that the invariant sets  $\tilde{U} := \bigcup_{n \in \mathbb{Z}} F_0^n(U)$  and  $\tilde{V} := \bigcup_{n \in \mathbb{Z}} F_0^n(V)$  are disjoint. As  $U$  is non-wandering we can assume, by going over to a suitable iterate, that  $F_0(U) \cap U \neq \emptyset$ . Thus, the successive iterates of  $U$  are attached to each other ‘like beads on a string’ and form an open and path-connected ‘tube’ which winds around the torus. As any open subset of  $U$  is also non-wandering, eventually this tube has to close, meaning that some higher iterate  $F_0^n(U)$  intersects  $U$  again after the tube has already performed a whole loop. Hence, the set  $\tilde{U}$  has the property that for each of its points it contains a closed curve going through this point and winding at least once around the torus.

By going over to an even higher iterate if necessary, we can repeat this argument and obtain the same property for  $\tilde{V}$ . But as both  $\tilde{U}$  and  $\tilde{V}$  are connected, this implies that one of the sets must lie strictly above the other, w.l.o.g.  $\varphi_{\tilde{U}}^+ \leq \varphi_{\tilde{V}}^-$ . Hence, as both sets are invariant the set  $[\varphi_{\tilde{U}}^+, \varphi_{\tilde{V}}^-]$  is an invariant strip, contradicting the assumption that there is no such strip between  $A$  and  $B$ .  $\square$

The following lemma allows to draw further conclusions from the existence of a wandering set:

**Lemma 4.4.** *Let  $A, B$  and  $C$  be as in Lemma 4.3 and suppose there is a wandering open set  $W \subseteq C$ . Then there exists a curve  $\gamma : \mathbb{T}^1 \rightarrow \mathbb{R}$  with  $\Gamma := \{(\theta, \gamma(\theta)) \mid \theta \in \mathbb{T}^1\} \subseteq C$  which is mapped either strictly above or strictly below itself by some iterate of  $F_0$ .*

*Proof.* We construct the point set  $\Gamma$  of the curve  $\gamma$ , as this will greatly simplify the argument. Note that we may allow  $\Gamma$  to contain vertical segments: The property we are interested in is open w.r.t. Hausdorff distance. Therefore any vertical parts can be slightly tilted in order to obtain a curve that can be represented as a graph over  $\mathbb{T}^1$ .

By going over to a suitable iterate, we can assume w.l.o.g. that  $W$  contains a ball of diameter larger than  $\omega$ . Let  $\Lambda_0 \subseteq W$  be a straight horizontal line segment of length  $\omega$ . Denote the endpoints of  $\Lambda_0$  by  $a_0 = (\theta_0, x_0)$  and  $b_0 = (\theta_0 + \omega, x_0)$ . Further, let  $\Lambda_k := F_0^k(\Lambda_0)$ ,  $a_k = (\theta_k, x_k) := F_0^k(a_0)$ ,  $b_k = (\theta_{k+1}, y_k) := F_0^k(b_0)$  and denote the closed vertical line segment between  $b_k$  and  $a_{k+1}$  by  $[b_k, a_{k+1}]$ , similarly for open and half-open segments. Finally, we define  $\Gamma_k := \bigcup_{i=0}^k \Lambda_i \cup \bigcup_{i=0}^{k-1} [b_i, a_{i+1}]$ . Note that thus  $\Gamma_k$  is a curve that joins  $a_0$  and  $b_k$  and contains  $k$  vertical segments. Further, by construction

$$(4.1) \quad F_0(\Gamma_k) \setminus \Gamma_k = \Lambda_{k+1} \cup (b_k a_{k+1}] .$$

Now let  $n := \max\{k \in \mathbb{N} \mid k\omega < 1\}$  and define  $\Gamma$  as the union of  $\Gamma_n \setminus \pi_1^{-1}([0, \omega))$ ,  $\Lambda_0$  and the vertical line segment between  $a_0$  and  $\Lambda_n$ . Note that this defines a closed curve that winds exactly once around the torus and contains  $n + 1$  vertical segments. Further, if we assume w.l.o.g.  $n \geq 2$ , then  $\Gamma_1 \subseteq \Gamma$ . From now on we assume that  $a_1$  lies above  $b_0$ , such that  $a_{k+1}$  lies above  $b_k$  for all  $k \in \mathbb{N}_0$  by monotonicity. (If  $a_1$  lies below  $b_0$ , this can be treated similarly.) We have to distinguish three different cases:

First, assume there exists  $m \in \mathbb{N}$  such that  $b_m$  lies above  $\Lambda_0$ .<sup>5</sup> W.l.o.g. we can assume  $m = n$ , otherwise we lift the system to the  $j$ -fold cover  $(\mathbb{R}/j\mathbb{Z}) \times \mathbb{T}^1$  of  $\mathbb{T}^2$ , where  $j$  is the integer

<sup>5</sup>Suppose  $\Lambda$  is the point set of a curve defined on a subinterval of  $\mathbb{T}^1$  and  $b \in \mathbb{T}^2$ . Then by saying  $b = (\theta, x)$  is above  $\Lambda$  we mean  $x > \sup \Lambda_\theta$ , implicitly assuming  $\pi_1(b) \in \pi_1(\Lambda)$ . In the following we will use similarly obvious terminology without further explanation.

part of  $m\omega$ , and repeat the construction of  $\Gamma$  as above. (It is obvious that if there exists a curve with the required property on this  $j$ -fold cover, then the same is true for the original system.) If  $b_n$  lies above  $\Lambda_0$ , by the assumption made above the same is true for  $a_{n+1}$ . Further, by the monotonicity of the fiber maps  $b_{n+1}$  lies above  $\Lambda_1$ . As  $\Lambda_{n+1}$  joins  $a_{n+1}$  and  $b_{n+1}$  and cannot intersect  $\Lambda_0 \cup \Lambda_1$  (recall that  $\Lambda_0$  is contained in the wandering set  $W$ ), we obtain that  $(b_n, a_{n+1}] \cup \Lambda_{n+1}$  lies strictly above the curve  $\Gamma_1$ . Together with (4.1) this implies  $F_0(\Gamma) \succ \Gamma$  and  $F_0^N(\Gamma) \succ \Gamma$  for suitably large  $N$ .

Secondly, assume there exists  $m \in \mathbb{N}$  such that  $b_m$  lies below  $\Lambda_0$  and  $\Lambda_{m+1}$  does not intersect  $[b_0, a_1]$ . Again, we can assume  $m = n$ . As  $\Lambda_{n+1}$  cannot intersect  $\Lambda_0 \cup \Lambda_1$ , it is disjoint from  $\Gamma_1$  in this case. But as  $b_{n+1}$  lies below  $\Lambda_1 \subseteq \Gamma_1$  by monotonicity, this implies that the whole curve  $\Lambda_{n+1} \cup (b_n, a_{n+1}]$  is below  $\Gamma_1$ . Similar to above we obtain  $F_0(\Gamma) \preccurlyeq \Gamma$  and  $F_0^N(\Gamma) \preccurlyeq \Gamma$  for suitably large  $N$ .

Finally, we have to address the case where  $b_m$  lies below  $\Lambda_0$  and  $\Lambda_{m+1}$  intersects  $[b_0, a_1]$  whenever  $\pi_1(b_m) \in \pi_1(\Lambda_0)$ . We show that this implies the existence of an invariant strip strictly between  $A$  and  $B$ , contradicting the assumption that there is no such strip.

Let  $\Omega := \bigcap_{k \geq 0} \bigcup_{j=k}^{\infty} \Gamma_j$ . Clearly  $\Omega$  is a non-empty, compact,  $F_0$ -invariant set, and consequently  $\varphi_{\Omega}^+$  is an upper semi-continuous invariant graph. We claim that

$$(4.2) \quad x_0 + \delta \leq \varphi_{\Omega}^+(\theta) \leq x_1 + \delta \quad \forall \theta \in I := (\theta_1 - \delta, \theta_1)$$

for a suitably small  $\delta > 0$ . This implies that the same inequalities hold for  $\varphi_{\Omega}^{+-}$ , such that  $[\varphi_{\Omega}^+, \varphi_{\Omega}^{+-}]$  defines an invariant strip which lies strictly between  $A$  and  $B$ . (Note that whenever a strip lies strictly between  $A$  and  $B$  on an open interval, then this is true on all of  $\mathbb{T}^1$ .)

As  $\Gamma_0$  is contained in the wandering set  $W$ , there exist small boxes  $W_0 := B_{\delta}(\theta_1) \times B_{\delta}(x_0)$  around  $b_0$  and  $W_1 := B_{\delta}(\theta_1) \times B_{\delta}(x_1)$  around  $a_1$  which no curve  $\Lambda_j$  with  $j \geq 2$  can intersect. We fix  $\delta \in (0, \omega)$  with this property. Now, whenever  $\pi_1(b_m) \in \pi_1(\Lambda_0)$  the curve  $\Lambda_m \cup [b_m, a_{m+1}] \cup \Lambda_{m+1} = F_0^m(\Gamma_1)$  has to pass through below the set  $\widehat{W}_1 := I \times B_{\delta}(x_1)$ : This holds for  $\Lambda_{m+1}$  as this curve must intersect  $[b_0, a_1]$  and cannot intersect  $\widehat{W}_1$ , and  $\Lambda_m$  lies below  $\Lambda_0$  anyway as this is true for its right endpoint  $b_m$ . (Recall here that we assumed that  $b_k$  is always below  $a_{k+1}$ , such that  $a_1$  is above  $\Lambda_0$ .) Consequently none of the sets  $\bigcup_{j \geq k} \Gamma_j$  intersects the region  $\{(\theta, x) \mid \theta \in I, x > x_1 - \delta\}$ , and from this the upper bound in (4.2) follows easily.

For the lower bound, note that there are infinitely many  $m \in \mathbb{N}$  such that  $\pi_1(W_1) \subseteq \pi_1(\Lambda_{m+1})$ . For these,  $\Lambda_{m+1}$  has to pass through between the boxes  $W_0$  and  $W_1$  on their whole width. Therefore the upper bounding graphs of the sets  $\bigcup_{j \geq k} \Gamma_j$  are always above  $W_0$ , and consequently the same is true for their pointwise limit  $\varphi_{\Omega}^+$ .  $\square$

*Proof of Proposition 4.1.* Fix  $f \in \mathcal{F}$  with lift  $F_0$  and suppose  $\rho(F_0) = 0$  and  $f$  has an invariant strip. Further, assume there exists no continuous curve  $\gamma^+ : \mathbb{T}^1 \rightarrow \mathbb{R}$  which is mapped strictly below itself by some iterate of  $F_0$ . We have to show that in this case there holds  $\rho(F_{\varepsilon}) > 0$  for all  $\varepsilon > 0$ . First of all, note that it is sufficient to prove that

$$(4.3) \quad \sup_{\theta, x, n} (F_{\varepsilon, \theta}^n(x) - x) = \infty \quad \forall \varepsilon > 0,$$

where the supremum is taken over all  $(\theta, x) \in \mathbb{T}^1 \times \mathbb{R}$  and  $n \in \mathbb{N}$ . Indeed, if  $\varepsilon' > 0$  is fixed we can apply (4.3) to  $\varepsilon'/2$  and obtain two possibilities: First, we could have  $\rho(F_{\varepsilon'/2}) > 0$ , but in this case we are finished as  $\rho(F_{\varepsilon'}) \geq \rho(F_{\varepsilon'/2}) > 0$ . The only other alternative is to have  $\rho(F_{\varepsilon'/2}) = 0$  but unbounded deviations for  $F_{\varepsilon'/2}$ . However, in this case  $\varepsilon \mapsto \rho(F_{\varepsilon})$  is strictly monotone in  $\varepsilon = \varepsilon'/2$  due to Lemma 3.1, such that again  $\rho(F_{\varepsilon'}) > 0$ .

Fix  $\varepsilon > 0$ . In order to prove (4.3), we will show that an orbit of  $F_{\varepsilon}$  moves upwards along a suitable sequence of minimal  $F_0$ -invariant strips  $A_n$ . As we want to proceed by induction and there might be uncountably many invariant strips, we have to make a certain choice in the construction of this sequence. To that end, given any minimal invariant strip  $A$  we denote by

$\mathcal{M}(A)$  the set of all minimal invariant strips  $B \succ A$ . (Note that  $\mathcal{M}(A)$  is always non-empty, as all integer translates of  $A$  are minimal invariant strips as well.) Further, if  $B \in \mathcal{M}(A)$ , let

$$D(A, B) := \inf_{\theta \in \mathbb{T}^1} (\varphi_B^-(\theta) - \varphi_A^+(\theta))$$

and

$$\mathcal{M}_\varepsilon(A) := \left\{ B \in \mathcal{M}(A) \mid D(A, B) \leq \frac{\varepsilon}{2} \right\}.$$

We start the construction of the sequence  $(A_n)_{n \in \mathbb{N}}$  with any minimal invariant strip  $A_0$ . (Note that 1, 1-invariant strips of  $f$  lift to  $F_0$ -invariant strips, so by assumption such an  $A_0$  always exists.) Then we proceed by induction in the following way: Suppose  $A_0 \prec A_1 \prec \dots \prec A_{n-1}$  have been chosen. We distinguish two cases:

(A1): If  $\mathcal{M}_\varepsilon(A_{n-1})$  is empty, we choose  $A_n$  as the (unique) minimal invariant strip above  $A_{n-1}$  with the property that there is no other invariant strip strictly between  $A_{n-1}$  and  $A_n$ .<sup>6</sup>

(A2): If  $\mathcal{M}_\varepsilon(A_{n-1})$  is non-empty we let

$$(4.4) \quad \psi(\theta) := \sup_{B \in \mathcal{M}_\varepsilon(A_{n-1})} \varphi_B^-(\theta)$$

and

$$(4.5) \quad A_n := [\psi^{+-}, \psi^+].$$

For sufficiently large  $c \in \mathbb{R}$ ,  $\psi$  is the lower bounding graph of the compact set

$$\mathcal{C} := \bigcap_{B \in \mathcal{M}_\varepsilon(A_{n-1})} [\varphi_B^-, c].$$

Thus  $\psi$  is l.s.c. and therefore  $A_n$  is a minimal invariant strip by Lemma 2.3. Further, it follows from the finite intersection property that  $\mathcal{C}$  intersects the set  $[\varphi_{A_{n-1}}^-, \varphi_{A_{n-1}}^+ + \varepsilon/2]$ . Hence, there exists at least one  $\theta' \in \mathbb{T}^1$  with

$$(4.6) \quad \psi(\theta') - \varphi_{A_{n-1}}^+(\theta') \leq \varepsilon/2.$$

Note that constructing the sequence  $A_n$  in this way we obtain  $D(A_{n-1}, A_{n+1}) \geq \varepsilon/2$  for all  $n \geq 1$ . Thus,

$$\lim_{n \rightarrow \infty} \inf_{\theta \in \mathbb{T}^1} \varphi_{A_n}^+(\theta) = \infty.$$

Therefore (4.3) is an immediate consequence of the following claim, which concludes the proof of the proposition:

**Claim:** For any  $n \geq 1$ ,  $\theta_0 \in \mathbb{T}^1$  and  $x_0 > \varphi_{A_{n-1}}^+(\theta_0)$  there exists  $k \in \mathbb{N}$  which satisfies

$$F_{\varepsilon, \theta_0}^k(x_0) > \varphi_{A_n}^+(\theta_0 + k\omega).$$

*Proof of the Claim.* First of all, we fix  $\delta_1 > 0$  such that

$$(4.7) \quad F_{0, \theta}^n(x) + \varepsilon/2 \leq F_{\varepsilon, \theta'}^n(x) \quad \forall \theta, \theta' : |\theta - \theta'| < \delta_1 \quad \forall x \in \mathbb{R}, \quad n \geq 1.$$

(Compare (3.1).) Further, we have to distinguish three cases. (See Lemmas 4.3 and 4.4 for the division of (A1) into the following two sub-cases.)

---

<sup>6</sup>  $A_n$  can be constructed as follows: Let  $\varphi(\theta) := \inf_{B \in \mathcal{M}(A_{n-1})} \varphi_B^+(\theta)$ . Then  $\varphi$  is an upper semi-continuous invariant graph, and  $A_n := [\varphi^-, \varphi^{++}]$  is a minimal invariant strip (see Lemma 2.3) which has the required property.

Case (A1)(a): *There is no invariant strip strictly between  $A_{n-1}$  and  $A_n$  and  $F_0$  restricted to  $(\varphi_{A_{n-1}}^+, \varphi_{A_n}^-)$  is topologically transitive.*

Choose  $\theta^*$  such that  $A_n$  is pinched at  $\theta^*$ , i.e.  $\varphi_{A_n}^-(\theta^*) = \varphi_{A_n}^+(\theta^*)$  (recall that  $A_n$  is pinched by Lemma 2.4). As  $A_n$  is compact we can further choose  $\delta_2 \in (0, \delta_1)$  such that

$$(4.8) \quad [\varphi_{A_n}^-(\theta), \varphi_{A_n}^+(\theta)] \subseteq B_{\varepsilon/8}(\varphi_{A_n}^+(\theta^*)) \quad \forall \theta \in B_{\delta_2}(\theta^*) .$$

Let  $(\theta', x')$  be a point with dense orbit in  $(\varphi_{A_{n-1}}^+, \varphi_{A_n}^-)$ ,  $\theta' \in B_{\delta_2/2}(\theta_0)$  and  $x' \leq x_0$ . Choose  $k \in \mathbb{N}$  such that  $\theta' + k\omega \in B_{\delta_2/2}(\theta^*)$  and  $F_{0,\theta'}^k(x') > \varphi_{A_n}^+(\theta^*) - \varepsilon/4$ . Then

$$F_{\varepsilon,\theta_0}^k(x_0) \geq F_{\varepsilon,\theta_0}^k(x') \stackrel{(4.7)}{\geq} F_{0,\theta'}^k(x') + \frac{\varepsilon}{2} > \varphi_{A_n}^+(\theta^*) + \frac{\varepsilon}{4} \stackrel{(4.8)}{>} \varphi_{A_n}^+(\theta_0 + k\omega) .$$

Case (A1)(b): *There is no invariant strip strictly between  $A_{n-1}$  and  $A_n$ , but there is a curve  $\gamma : \mathbb{T}^1 \rightarrow \mathbb{R}$  which is mapped strictly above itself by some iterate of  $F_0$ .*

Define  $\gamma_n(\theta) := F_{\theta-n\omega}^n(\gamma(\theta - n\omega))$ . As this sequence of continuous curves is monotonically increasing and bounded, it must converge pointwise to some l.s.c. invariant graph  $\gamma_\infty := \lim_{n \rightarrow \infty} \gamma_n$ . Note that  $\gamma_\infty$  does not necessarily have to coincide with  $\varphi_{A_n}^-$ . However, we must have  $\gamma_\infty^+ = \varphi_{A_n}^+$ , otherwise  $[\gamma_\infty, \gamma_\infty^+]$  would be an invariant strip strictly between  $A_{n-1}$  and  $A_n$ . Therefore, by Lemma 2.4 the set  $\tilde{A} := [\gamma_\infty, \varphi_{A_n}^+]$  is pinched. Similarly,  $\gamma_{-\infty} := \lim_{n \rightarrow \infty} \gamma_{-n}$  defines an u.s.c. invariant graph which is pinched to  $\varphi_{A_{n-1}}^-$ .

Let  $\theta^*$  be a fiber on which  $\tilde{A}$  is pinched. Due to the compactness of  $\tilde{A}$  we can choose  $\delta_2 \in (0, \delta_1)$  for which

$$(4.9) \quad [\gamma_\infty(\theta), \varphi_{A_n}^+(\theta)] \subseteq B_{\varepsilon/8}(\varphi_{A_n}^+(\theta^*)) \quad \forall \theta \in B_{\delta_2}(\theta^*) .$$

Fix  $k_1 \in \mathbb{N}$  such that for some  $\theta' \in B_{\delta_2/2}(\theta_0)$  there holds  $\gamma_{-k_1}(\theta') \leq x_0$ . Note that such  $k_1$  and  $\theta'$  exist even if  $x_0 < \gamma_{-\infty}(\theta_0)$ , because arbitrarily close to  $\theta_0$  there are fibers where  $[\varphi_{A_{n-1}}^-, \gamma_{-\infty}]$  is pinched. Further, fix  $K$  and  $\delta_3 \in (0, \delta_2/2)$  with the property that

$$(4.10) \quad \gamma_j(\theta) \geq \varphi_{A_n}^+(\theta^*) - \varepsilon/4 \quad \forall j \geq K, \theta \in B_{\delta_3}(\theta^*) .$$

Choose  $k_2 \geq K$  such that  $\theta' + (k_1 + k_2)\omega \in B_{\delta_3}(\theta^*)$  and let  $k := k_1 + k_2$ . Then

$$\begin{aligned} F_{\varepsilon,\theta_0}^k(x_0) &\stackrel{(4.7)}{\geq} F_{0,\theta'}^k(x_0) + \frac{\varepsilon}{2} \geq F_{0,\theta'}^k(\gamma_{-k_1}(\theta')) + \frac{\varepsilon}{2} = \gamma_{k_2}(\theta' + k\omega) + \frac{\varepsilon}{2} \\ &\stackrel{(4.10)}{\geq} \varphi_{A_n}^+(\theta^*) + \frac{\varepsilon}{4} \stackrel{(4.9)}{>} \varphi_{A_n}^+(\theta_0 + k\omega) . \end{aligned}$$

Case (A2):  $A_n$  is defined by (4.4) and (4.5).

By definition (4.5) we have  $\varphi_{A_n}^+ = \psi^+$ , and hence  $\hat{A} := [\psi, \varphi_{A_n}^+]$  is pinched by Lemma 2.4 (recall that  $\psi$  is l.s.c. by definition). As before, we choose some fiber  $\theta^*$  on which  $\hat{A}$  is pinched and fix  $\delta_2 \in (0, \delta_1)$  such that

$$(4.11) \quad [\psi(\theta), \varphi_{A_n}^+(\theta)] \subseteq B_{\varepsilon/8}(\varphi_{A_n}^+(\theta^*)) \quad \forall \theta \in B_{\delta_2}(\theta^*) .$$

Further, we choose  $\theta' \in \mathbb{T}^1$  which satisfies  $\varphi_{A_{n-1}}^+(\theta') \geq \psi(\theta') - \varepsilon/2$  (see (4.6)). By Lemma 2.3 we have  $\varphi_{A_{n-1}}^+ = \varphi_{A_{n-1}}^{-+}$  and therefore  $(\theta', \varphi_{A_{n-1}}^+(\theta')) \in \overline{\Phi_{A_{n-1}}^-}$ . Consequently, due to the l.s.c. of  $\varphi_{A_{n-1}}^-$ , there exists an open set  $U \subseteq B_{\delta_2/2}(\theta')$  such that

$$(4.12) \quad \varphi_{A_{n-1}}^+(\theta) \geq \varphi_{A_{n-1}}^-(\theta) \geq \psi(\theta') - \varepsilon \quad \forall \theta \in U .$$

Choose  $k_1 \geq 1$  for which  $\theta_0 + k_1\omega \in U$ . Then

$$(4.13) \quad F_{\varepsilon, \theta_0}^{k_1}(x_0) \geq F_{0, \theta_0}^{k_1}(x_0) + \varepsilon \geq \varphi_{A_{n-1}}^+(\theta_0 + k_1\omega) + \varepsilon \stackrel{(4.12)}{\geq} \psi(\theta').$$

Now let  $k_2 \geq 1$  such that  $\theta' + k_2\omega \in B_{\delta_2/2}(\theta^*)$  and define  $k := k_1 + k_2$ . Then

$$\begin{aligned} F_{\varepsilon, \theta_0}^k(x_0) &= F_{\varepsilon, \theta_0 + k_1\omega}^{k_2}(F_{\varepsilon, \theta_0}^{k_1}(x_0)) \stackrel{(4.13)}{\geq} F_{\varepsilon, \theta_0 + k_1\omega}^{k_2}(\psi(\theta')) \\ &\stackrel{(4.7)}{\geq} F_{0, \theta'}^{k_2}(\psi(\theta')) + \frac{\varepsilon}{2} = \psi(\theta' + k_2\omega) + \frac{\varepsilon}{2} \stackrel{(4.11)}{>} \varphi_{A_n}^+(\theta_0 + k\omega). \end{aligned}$$

□

## References

- [1] M. Herman. Une méthode pour minorer les exposants de Lyapunov et quelques exemples montrant le caractère local d'un théorème d'Arnold et de Moser sur le tore de dimension 2. *Commentarii Mathematici Helvetici*, 58:453–502, 1983.
- [2] T. Jäger and G. Keller. The Denjoy type-of argument for quasiperiodically forced circle diffeomorphisms. *Ergodic Theory and Dynamical Systems*, 26(2):447–465, 2006.
- [3] T. Jäger and J. Stark. Towards a classification of quasiperiodically forced circle homeomorphisms. *Journal of the LMS*, 73(3):727–244, 2005.
- [4] F. Delyon and B. Souillard. The rotation number for finite difference operators and its properties. *Communucations in Mathematical Physics*, 89:415–426, 1983.
- [5] R. Johnson and J. Moser. The rotation numner for almost periodic potentials. *Communications in Mathematical Physics*, 84:403–438, 1982.
- [6] A. Prasad, S. Negi, and R. Ramaswamy. Strange nonchaotic attractors. *International Journal of Bifurcation and Chaos*, 11(2):291–309, 2001.
- [7] M. Ding, C. Grebogi and E. Ott. Evolution of attractors in quasiperiodically forced systems: From quasiperiodic to strange nonchaotic to chaotic. *Physical Review A*, 39(5):2593–2598, 1989.
- [8] U. Feudel, J. Kurths and A. Pikovsky. Strange nonchaotic attractor in a quasiperiodically forced circle map. *Physica D*, 88:176–186, 1995.
- [9] P. Glendinning, U. Feudel, A.S. Pikovsky, and J. Stark. The structure of mode-locked regions in quasi-periodically forced circle maps. *Physica D*, 140:227–243, 2000.
- [10] J. Stark, U. Feudel, P. Glendinning, and A. Pikovsky. Rotation numbers for quasiperiodically forced monotone circle maps. *Dynamical Systems*, 17(1):1–28, 2002.
- [11] K. Bjerklöv and T. Jäger. Strange non-chaotic attractors in quasiperiodically forced circle maps. In preparation.
- [12] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, 1997.
- [13] K. Bjerklöv. Positive Lyapunov exponent and minimality for a class of one-dimensional quasi-periodic Schrödinger equations. *Ergodic Theory and Dynamical Systems*, 25:1015–1045, 2005.
- [14] J. Stark. Transitive sets for quasiperiodically forced monotone maps. *Dynamical Systems*, 18(4):351–364, 2003.
- [15] H. Furstenberg. Strict ergodicity and transformation of the torus. *American Journal of Mathematics*, 83:573–601, 1961.
- [16] L. Arnold. *Random Dynamical Systems*. Springer, 1998.